

Handling vacuum regions in a hybrid plasma solver

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Abstract

In a hybrid plasma solver (particle ions, fluid mass-less electrons) regions of vacuum, or very low charge density, can cause problems since the evaluation of the electric field involves division by charge density. This causes large electric fields in low density regions that can lead to numerical instabilities. Here we propose a self consistent handling of vacuum regions for hybrid solvers.

Vacuum regions can be considered having infinite resistivity, and in this limit Faraday's law approaches a magnetic diffusion equation. We describe an algorithm that solves such a diffusion equation in regions with charge density below a threshold value. We also present an implementation of this algorithm in a hybrid plasma solver, and an application to the interaction between the Moon and the solar wind.

We also discuss the implementation of hyperresistivity for smoothing the electric field in a PIC solver.

1 The hybrid equations

In the hybrid approximation, ions are treated as particles, and electrons as a massless fluid. In what follows we use SI units. The trajectory of an ion, $\mathbf{r}(t)$ and $\mathbf{v}(t)$, with charge q and mass m , is computed from the Lorentz force,

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

where $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ is the electric field, and $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ is the magnetic field. From now on we do not write out the dependence on \mathbf{r} and t . The electric field is given by

$$\mathbf{E} = \frac{1}{\rho_I} (-\mathbf{J}_I \times \mathbf{B} + \mu_0^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p_e) + \frac{\eta}{\mu_0} \nabla \times \mathbf{B}, \quad (1)$$

where ρ_I is the ion charge density, \mathbf{J}_I is the ion current density, p_e is the electron pressure, η is the resistivity, and $\mu_0 = 4\pi \cdot 10^{-7}$ is the magnetic constant. Then Faraday's law is used to advance the magnetic field in time,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \quad (2)$$

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Note that the unknowns are the position and velocity of the ions, and the magnetic field on a grid, *not* the electric field, since it can always be computed from (1). Further details on the hybrid model used here, and the discretization, can be found in Holmström (2011a,b).

2 Vacuum regions

In regions of low ion charge density, ρ_I , the hybrid method can have numerical problems. We see from (1) that the electric field computation involves a division by ρ_I . In what follows we will not write out the I subscript, i.e. $\rho = \rho_I$. Thus, in low density regions we will have large electric fields, and in the limit of zero charge density, the electric field magnitude will tend to infinity. This can lead to numerical instabilities, due to large gradients in the electric field, and due to large accelerations of ions. The solution quickly becomes unstable.

We can either have a region where $\rho = 0$ physically, like inside a resistive obstacle, or due to the statistical nature of particle in cell solvers, there can be regions where we simply have no macro particles, or very few. From now on we denote all the different cases of low density regions as *vacuum regions*.

Many ad hoc solutions have been proposed to handle vacuum regions. One way is to set a minimum allowed ion charge density, e.g., if ρ is below a threshold value in a cell, ρ is set to that value for the cell. A similar solution is to set a maximum allowed value for the electric field. One can also introduce new ion sources that were not part of the original problem, to keep ρ large enough in all cells. If we have an absorbing obstacle we can instead of removing absorbed ions reduce their weight gradually over time toward zero. None of these solutions solve the original problem. In the case of threshold values we do not get the solution to the hybrid equations, and in the case of artificial sources or losses, we are solving the equations self consistently, but we are solving a different physical problem.

However, a self consistent, physically correct way of handling the problem of vacuum regions was proposed a long time ago. Hewett (1980) noted that vacuum regions can be viewed as having infinite resistivity, and an algorithm can be devised where the resistivity in (1) is set to a large value in regions of low density. Harned (1982) also used a similar idea and solved a Laplace equation for the electric field in vacuum regions. However, solving Laplace equation over a complicated region that is also changing over time is not an easy task. Especially if one wants to do it on a parallel computer, since solving Laplace equation on a grid involves solving a system of linear equations.

3 Solving a diffusion equation in vacuum regions

Building on the idea of Hewett (1980) of having a large resistivity in low density regions, let us see what a large resistivity implies.

Let the resistivity be variable in space, and time, $\eta = \eta(\mathbf{r}, t)$. If we assume that the resistive term dominate in the expression (1) for the electric field, then Faraday's law (2) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right).$$

For a constant resistivity we have that

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}. \quad (3)$$

This is a diffusion equation for the magnetic field, and the steady state solution will be a solution to the Laplace equation $\nabla^2 \mathbf{B} = 0$.

4 Time step limits for the diffusion equation

The diffusion equation for the magnetic field (3) is similar to the heat equation and gives a time step limit for stability of

$$\Delta t < \frac{\mu_0 \Delta x^2}{2\eta},$$

where Δt is the time step for an explicit time integrator. For large resistivities this will set the limit for the allowed length of the time step, e.g., if we increase η by 10, Δt needs to be decreased by 10 for stability.

5 Algorithm

The details of the algorithm for handling low density regions is as follows. We assume that we have a resistivity, $\eta = \eta(\mathbf{r}, t)$, that is defined in all regions of the solution domain. A cell is denoted a *vacuum cell* if $\rho < \rho_v$ for the cell. The electric field in a vacuum cell is computed by (1), with $1/\rho = 0$, i.e. only the resistive term is non-zero. If the vacuum cell is outside any obstacle, then we set the resistivity to a vacuum resistivity, $\eta = \eta_v$. If the vacuum cell is in an obstacle, we keep the original resistivity.

The advantage of such an approach is that vacuum regions and regions of a resistive object can be handled by the original solver. The time advance of Faraday's law (2) is the same for all cells. We only compute the electric field differently in different cells.

6 Hyperresistivity

The leapfrog time stepping scheme that we use to discretize Faraday's law in time has no numerical diffusivity. To stabilize the computations one may have to smooth the solution. Here we choose to use a hyperresistivity term in the electric field computation.

Hyperresistivity of different orders were introduced by Maron (2008) for an MHD solver. Here we introduce a hyperresistivity, η_h , and the expression for the electric field (1) becomes

$$\mathbf{E} = \frac{1}{\rho_I} (-\mathbf{J}_I \times \mathbf{B} + \mathbf{J} \times \mathbf{B} - \nabla p_e) + \eta \mathbf{J} - \eta_h \nabla^2 \mathbf{J}, \quad (4)$$

where the current, $\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$. The advantage of hyperresistivity compared to smoothing, that is often implemented in hybrid solvers, is that the higher order derivatives of the Laplacian diffuse high frequency structures, while preserving lower frequency ones (Maron, 2008). Also, the addition of hyperresistivity

can often increase the maximum stable time step. In our implementation, the Laplacian

$$\nabla^2 \mathbf{J} = \left(\frac{\partial^2 J_x}{\partial x^2} \frac{\partial^2 J_y}{\partial y^2} \frac{\partial^2 J_z}{\partial z^2} \right)^T$$

is discretized using standard second order finite difference stencils. The hyper-resistive term is present in all parts of the domain, also in the vacuum regions, and in the examples that follows, we have used a value of $\eta_h = 5 \cdot 10^{14}$.

7 An application: The Lunar plasma wake

A good example where vacuum and low density regions occur is the plasma wake behind the Moon, formed by the absorption on the dayside of the impinging solar wind plasma. More details on a hybrid model of the interaction between the Moon and the solar wind can be found in Holmström (2012). The simulation parameters used here are similar to those in that work, with an interplanetary magnetic field (IMF) at a 45° angle to the solar wind flow.

The different resistivity regions at $t = 30$ s are shown in Fig. 1. Here η is defined to be 0, except for the interior of the Moon (a sphere of radius 1730 km) where $\eta = 10^7 \Omega\text{m}$. Cells with a relative ion charge density less than 0.0001, that are outside the obstacle (the Moon), was considered vacuum cell ($\rho_v = 0.0001$), and there the resistivity was set to a vacuum resistivity $\eta_v = 10^8$. We clearly see that the vacuum region is irregular. It also changes with time.

By following the above approach to handle vacuum regions, we have introduced two new numerical parameters. A threshold ion charge density, ρ_v , below which we consider a cell to be a vacuum cell; and a vacuum resistivity, η_v , that we set the resistivity to in such cells. Ideally, η_v should be as large as possible and ρ_v should be as small as possible, for the solution to approach the solution to the original hybrid equations. What limits the value of η_v is the time step limit discussed in Section 4. The computational time will increase in proportion to η_v for an explicit time integrator. The effect of different minimum density, ρ_v , are shown in Fig. 2. We see that ρ_v has to be small enough to get a smooth transition between vacuum and non-vacuum regions. The solution converges as ρ_v is decreased, as can be seen especially in the central wake region. We can note that the relative ρ_v is as small as one part in 10000 for the converged solution. This can be contrasted with the approach of just setting a minimum charge density. Then one typically has to choose a value of the relative density on the order of a few %, otherwise numerical instabilities will develop. In contrast, the magnetic diffusion equation has to be applied only in cells of very low density. The method is also stable in the presence of large gradients in the resistivity. The example shown in Fig. 1 and 2 has a step change in resistivity, from 0 to 10^7 at the Lunar surface.

8 Summary and conclusions

We have shown that vacuum regions in a plasma particle in cell solver can be handled by setting a high resistivity in those regions. This leads to the solution of a magnetic diffusion equation in such regions. A minimum charge density parameter decide what cells are vacuum, where the magnetic diffusion equation

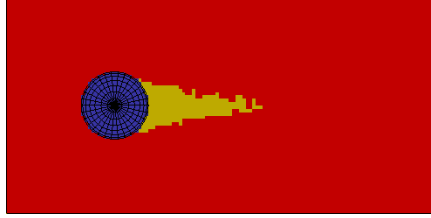


Figure 1: Resistivity in different regions of the simulation domain at $t = 30$ s. A cut in the plane of the IMF. The solar wind flows in from the left. The blue region is the interior of the Moon with $\eta = 10^7 \Omega\text{m}$. Red shows solar wind plasma with $\eta = 0$, and in yellow is the wake region with $\eta_v = 10^8$, where $\rho < \rho_v = 0.0001$.

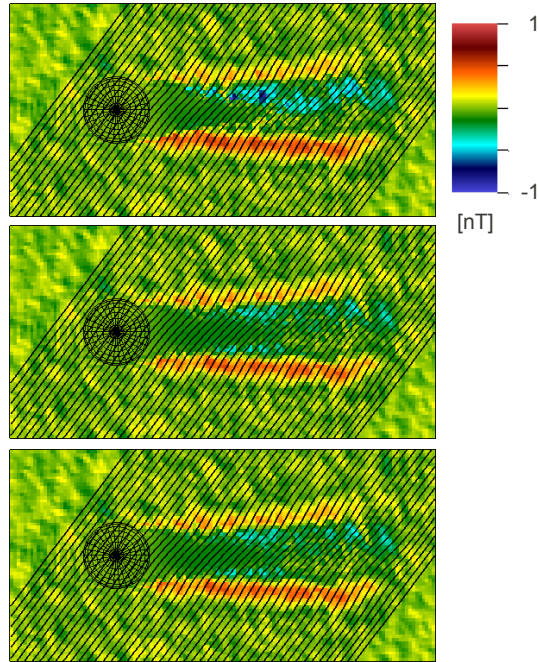


Figure 2: The effect of different minimum charge densities, ρ_v . Shown is the component of the magnetic field that is perpendicular to the plane containing the IMF. The minimum charge density, ρ_v , decrease from top to bottom, with values of 0.01, 0.001, and 0.0001. The cuts are planes that contain the IMF, and the black lines are the magnetic field lines. The solar wind flows in from the left.

should be advanced in time. It is also possible to include arbitrary resistive obstacles. The algorithm was exemplified by modeling the plasma interaction between the solar wind and the Moon.

The advantage of the method is that it handles vacuum regions and resistive obstacles in a self consistent manner. Also, the magnetic diffusion equation only has to be applied in very low density cells, e.g., a relative density of 0.0001, compared to a reference density. The method seems stable to discontinuities in the resistivity.

A disadvantage is that the time advance of the magnetic diffusion equation requires a small time step. This is however needed anyway for some problems, e.g., if we model the Moon and want to include crustal magnetic anomalies. A solution for future study would be to use an implicit time integrator to advance Faraday's law.

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